Programming with (co-)inductive types in Coq

Matthieu Sozeau

February 3rd 2014

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Last time

1. Record Types
2. Mathematical Structures and Coercions
3. Type Classes and Canonical Structures
4. Interfaces and Implementations
5. **TODO** Monadic Programming with Type Classes

Questions?

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Today

In this class, we shall present how Coq allows us in practice to define data types using (co-)inductive declarations, compute on these datatypes, and reason by induction.

Inductive declarations

An arbitrary type as assumed by:

```
Variable T : Type.
```

gives no a priori information on the nature, the number, or the properties of its inhabitants.

Inductive declarations

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Inductive bool : Set := true : bool | false : bool.

```
Print nat.
```

Inductive nat : Set := O : nat | S : nat -> nat.

Each such rule is called a constructor.

Enumerated types

Enumerated types are types which list and name exhaustively their inhabitants.

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```
Inductive bool : Set := true : bool | false : bool.

Inductive color:Type :=
| white | black | yellow | cyan | magenta
| red | blue | green.

Check cyan.
```

cyan : color

Labels refer to distinct elements.

Enumerated types: program by case analysis

Inspect the enumerated type inhabitants and assign values:

```
Definition my_negb (b : bool) :=
  match b with true => false | false => true.
```

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```
Definition my_negb (b : bool) :=
  match b with true => false | false => true.

Definition is_black_or_white (x : color) : bool :=
  match x with
  | black => true
  | white => true
  | _ => false
  end.
```

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Compute: constructors are values.

```
Eval compute in (is_black_or_white hat).
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Compute: constructors are values.

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Eval compute in (is_black_or_white hat).
                                         = false
                                         : bool
```

Enumerated types: reason by case analysis

Inspect the enumerated type inhabitants and build proofs:

```
Lemma bool_case : forall b : bool, b = true \/ b = false.
Proof.
intro b.
case b.
  left; reflexivity.
right; reflexivity.
Qed.
```

Enumerated types: reason by case analysis

Inspect the enumerated type inhabitants and build proofs:

```
Lemma is_black_or_whiteP : forall x : color,
  is_black_or_white x = true ->
  x = black \/ x = white.
Proof.
(* Case analysis + computation *)
intro x; case x; simpl; intro e.
(* In the three first cases: e: false = true *)
  discriminate e.
  discriminate e.
  discriminate e.
(* Now: e: true = true *)
  left; reflexivity.
  right; reflexivity.
Qed.
```

Enumerated types: reason by case analysis

Two important tactics, not specific to enumerated types:

- `simpl`: makes computation progress (pattern matching applied to a term starting with a constructor)
- `discriminate`: allows to use the fact that constructors are distincts:
  - `discriminate H`: closes a goal featuring a hypothesis `H` like `(H : true = false)`;
  - `discriminate`: closes a goal like `(O <> S n)`.

Options and partial functions

Function $f : A \to B$ defined on only a subdomain D of A.

- Return a default value in B for $x \notin D$
  Arbitrary if B is a variable : head of list
- Modify the return type: `option` B.

```
Inductive option:Type :=
  Some : B -> option | None : option.
```

  - The program tests whether the input is inside the domain
  - Similar to exceptions
  - $\forall x, D\,x \Rightarrow g\,x = \texttt{Some}\,(f\,x)$.
- Extra argument of domain: $\forall x, x \in D \to B$
  - Argument erased by extraction: $D : A \to Prop$.
  - Proof irrelevance : $f\,x\,d_1 = f\,x\,d_2$

Recursive types

Let us craft new inductive types:

```
Inductive natBinTree : Set :=
```

Recursive types

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| Leaf : nat -> natBinTree
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Recursive types

Let us craft new inductive types:

```
Inductive natBinTree : Set :=
| Leaf : nat -> natBinTree
| Node : nat -> natBinTree -> natBinTree -> natBinTree.
```

```
Inductive term : Set :=
|Zero : term
|One : term
|Plus : term -> term -> term
|Mult : term -> term -> term.
```

An inhabitant of a recursive type is built from a finite number of constructor applications.

Recursive types: program by case analysis

We have already seen some examples of such pattern matching:

```
Definition isNotTwo x :=
  match x with
  | S (S O) => false
  | _ => true
end.
```

Recursive types: program by case analysis

We have already seen some examples of such pattern matching:

```
Definition isNotTwo x :=
  match x with
  | S (S O) => false
  | _ => true
end.

Definition is_single_nBT (t : natBinTree) :=
match t with
|Leaf _ => true
|_ => false
end.
```

Recursive types: proofs by case analysis

```
Lemma is_single_nBTP : forall t,
  is_single_nBT t = true -> exists n : nat, t = Leaf n.
Proof.
```

Recursive types: proofs by case analysis

```
Lemma is_single_nBTP : forall t,
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Proof.
(* We use the possibility to destruct the tree
  while introducing *)
intros [ nleaf | nnode t1 t2] h.
```

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(* First case: we use the available label *)
  exists nleaf.
  reflexivity.
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(* Second case: the test evaluates to false *)
simpl in h.
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  reflexivity.
(* Second case: the test evaluates to false *)
simpl in h.
discriminate.
Qed.
```

Recursive types

Constructors are injective:

```
  Lemma inj_leaf : forall x y, Leaf x = Leaf y -> x = y.
  Proof.
  intros x y hLxLy.
  injection hLxLy.
  trivial.
Qed.
```

Recursive types: structural induction

Let us go back to the definition of natural numbers:

```
Inductive nat : Set := O : nat | S : nat -> nat.
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The Inductive keyword means that at definition time, this system geneates an induction principle:

```
nat_ind
     : forall P : nat -> Prop,
       P 0 ->
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```
Inductive nat : Set := O : nat | S : nat -> nat.
```

The Inductive keyword means that at definition time, this system geneates an induction principle:

```
nat_ind
     : forall P : nat -> Prop,
       P 0 ->
       (forall n : nat, P n -> P (S n)) ->
       forall n : nat, P n
```

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To prove that for `P : term -> Prop`, the theorem `forall t : term, P t` holds, it is sufficient to:

Recursive types: structural induction

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- Prove that the property holds for the base cases:
  - `(P Zero)`
  - `(P One)`

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To prove that for `P : term -> Prop`, the theorem `forall t : term, P t` holds, it is sufficient to:

- Prove that the property holds for the base cases:
  - `(P Zero)`
  - `(P One)`
- Prove that the property is transmitted inductively:
  - `forall t1 t2 : term,`
    `P t1 -> P t2 -> P (Plus t1 t2)`
  - `forall t1 t2 : term,`
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Recursive types: structural induction

To prove that for `P : term -> Prop`, the theorem `forall t : term, P t` holds, it is sufficient to:

- Prove that the property holds for the base cases:
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- Prove that the property is transmitted inductively:
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    `P t1 -> P t2 -> P (Plus t1 t2)`
  - `forall t1 t2 : term,`
    `P t1 -> P t2 -> P (Mult t1 t2)`

The type `term` is the smallest type containing `Zero` and `One`, and closed under `Plus` and `Mult`.

Recursive types: structural induction

The induction principles generated at definition time by the system allow to:

- Program by recursion (`Fixpoint`)
- Prove by induction (`induction`)

Recursive types: program by structural induction

We can compute some information on the size of a term:

```
Fixpoint height (t : natBinTree) : nat :=
  match t with
    |Leaf _ => 0
    |Node _ t1 t2 => Max.max (height t1) (height t2) + 1
  end.
```

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  end.

Fixpoint size (t : natBinTree) : nat :=
  match t with
```

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  end.

Fixpoint size (t : natBinTree) : nat :=
  match t with
  |Leaf _ => 1
```

Recursive types: program by structural induction

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    |Node _ t1 t2 => Max.max (height t1) (height t2) + 1
  end.

Fixpoint size (t : natBinTree) : nat :=
  match t with
  |Leaf _ => 1
  |Node _ t1 t2 => (size t1) + (size t2) + 1
  end.
```

Recursive types: program by structural induction

We can access some information contained in a term:

```
Require Import List.
Fixpoint label_at_occ (dflt : nat)
               (t : natBinTree)(u : list bool) :=
match u, t with
|nil, _ =>
  (match t with Leaf n => n | Node n _ _ => n end)
|b :: tl, t =>
  match t with
    |Leaf _ => dflt
    | Node _ t1 t2 =>
      if b then label_at_occ dflt t2 tl
      else label_at_occ dflt t1 tl
  end
end.
```

Recursive types: proofs by structural induction

We have already seen induction at work on nats and lists.
Here its goes on binary trees:

```
Lemma le_height_size : forall t : natBinTree,
           height t <= size t.
Proof.
induction t; simpl.
  auto.
apply plus_le_compat_r.
apply max_case.
  apply (le_trans _ _  _ IHt1).
  apply le_plus_l.
  apply (le_trans _ _  _ IHt2).
  apply le_plus_r.
Qed.
```

Structure of the definition of a recursive function

```
Inductive btree : Type := Leaf : btree
                 | Node : btree -> btree -> btree.

Fixpoint get_subtree
   (l:list bool) (t:btree) {struct t} : btree :=
  match t, l with
  | Empty, _ => Empty
  | Node _ _, nil => t
  | Node tl tr, b :: l' =>
    if b then get_subtree l' tl else get_subtree l' tr
  end.
```

- Note the recursive calls made on `tl` and `tr`.
- The recursive call should be done on a strict sub-term of the argument.
- This ensures the termination of recursive functions

Termination

The termination of recursive functions is one of the component which ensures the logical consistency of Coq.

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We have to live with this . . .

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We have to live with this ...

And we have to convince the system that all the functions we write are terminating.

An example of recursive function: fact

Recursive call should be made on strict sub-term:

```
Fixpoint fact n :=
  match n with
  | O => 1
  | S n' => n * fact n'
  end.

Definition fact' :=
  fix fact1 n :=
    match n with
    | O => 1
    | S n' => n * fact1 n'
    end.
```

An example of recursive function: div2

Recursive call can be done on not immediate sub-terms:

```
Fixpoint div2 n :=
  match n with
  | S (S n’) => S (div2 n’)
  | _ => 0
  end.
```

A sub-term of strict sub-term is a strict sub-term

More general recursive calls

- It is possible to have recursive calls on results of functions.
- All cases must return a strict sub-term.
- Strict sub-terms may be obtained by applying functions on strict sub-terms.
  - This functions should only return sub-terms of their arguments. (not necessarily strict ones).
  - The system checks by looking at all cases.

Example of function that returns a sub-term

```
Definition pred (n : nat) :=
  match n with
  | O =>  n
  | S p => p
  end.
```

- In the `O` branch, the value is `n`, a (non-strict) sub-term of `n`.
- In the `S p` branch, the value is `n` a (strict) sub-term of `n`.

Recursive function using `pred`

```
Fixpoint div2' (n : nat) :=
  match n with
    O => n
  | S p => S (div2' (pred p))
  end.
```

The same trick can be played with `minus` which returns a sub-term of its first argument, to define euclidian division.

Mutual recursion

It is possible to define function by mutual recursion:

```
Fixpoint even n :=
  match n with
  | O => true
  | S n' => odd n'
  end
with odd n :=
  match n with
  | O => false
  | S n' => even n'
  end.
```

Lexicographic order

Sometimes termination functions is ensured by a lexicographic order on arguments. In Ocaml we can program:

```
let rec merge l1 l2 =
  match l1, l2 with
  | [], _ -> l2
  | _, [] -> l1
  | x1::l1', x2::l2' ->
    if x1 <= x2 then
      x1 :: merge l1' l2
    else
      x2 :: merge l1 l2';;
```

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      x1 :: merge l1' l2
    else
      x2 :: merge l1 l2';;
```

There are two recursive calls `merge l1' l2` and `merge l1 l2'`.

Solution in Coq: internal recursion

Coq also makes it possible to describe *anonymous* recursive function
Sometimes necessary to use them for difficult recursion patterns

```
Fixpoint merge (l1 l2:list nat) : list nat :=
  match l1, l2 with
  | nil, _ => l2 |  _, nil => l1
  | x1::l1', x2::l2' =>
    if leb x1 x2 then x1::merge l1' l2
    else
      x2 :: (fix merge_aux (l2:list nat) :=
              match l2 with
              | nil => l1
              | x2::l2' =>
                if leb x1 x2 then x1::merge l1' l2
                else x2:: merge_aux l2'
              end) l2'
  end.
```

The style is not very readable (use the `Section` instead)

Another solution (Hugo Herbelin)

```
Fixpoint merge l1 l2 :=
  let fix merge_aux l2 :=
   match l1, l2 with
   | nil, _ => l2
   | _, nil => l1
   | x1::l1', x2::l2' =>
     if leb x1 x2 then x1::merge l1' l2
     else x2::merge_aux l2'
   end
  in merge_aux l2.

Compute merge (2::3::5::7::nil) (3::4::10::nil).
= 2 :: 3 :: 3 :: 4 :: 5 :: 7 :: 10 :: nil
     : list nat
```

More general recursion

- Constraints of structural recursion may be too cumbersome.
- Sometimes a measure decreases, which cannot be expressed by structural recursion.
- The general solution provided by *well-founded* recursion.
- An intermediate solution provided by the command `Function`.

Example using `Function`: fact on Z

Integers have a more complex structure than natural numbers

```
Inductive positive : Set :=
  | xH : positive                  (* encoding of 1     *)
  | xO : positive -> positive      (* encoding of 2*p   *)
  | xI : positive -> positive.     (* encoding of 2*p+1 *)

Inductive Z : Set :=
  | Z0: Z | Zpos: positive -> Z | Zneg: positive -> Z.
```

- This type makes computation more efficient.
- $x - 1$ is not a structural sub-term of x.
- For instance 3 is `Zpos (xI xH)` and 2 is `Zpos (xO xH)`.

Example using Function: fact on Z

```
Require Import Recdef.

Function factZ (x : Z) {measure Zabs_nat x} :=
  if Zle_bool x 0 then 1 else x * fact (x - 1).
1 subgoal
  ============================
   forall x : Z, Zle_bool x 0 = false ->
    (Zabs_nat (x - 1) < Zabs_nat x)%nat
```

Now, we prove explicitely that the measure decreases.

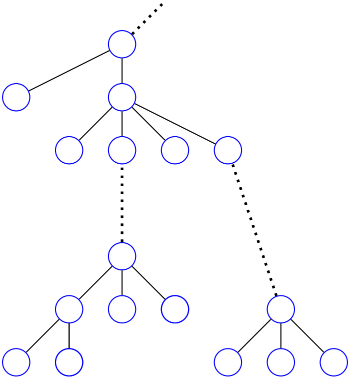
Merge again

```
Definition slen (p:list nat * list nat) :=
  length (fst p) + length (snd p).

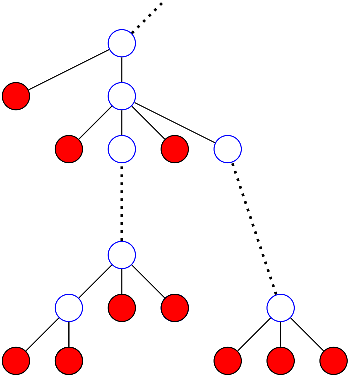
Function Merge (p:list nat * list nat)
   { measure slen p } : list nat :=
  match p with
  | (nil, l2) => l2
  | (l1, nil) => l1
  | ((x1::l1') as l1, (x2::l2') as l2) =>
     if leb x1 x2 then x1::Merge (l1',l2)
     else x2::Merge (l1,l2')
   end.
(* Two goals *)
...
Defined.

Compute Merge (2::3::5::7::nil, 3::4::10::nil).
```

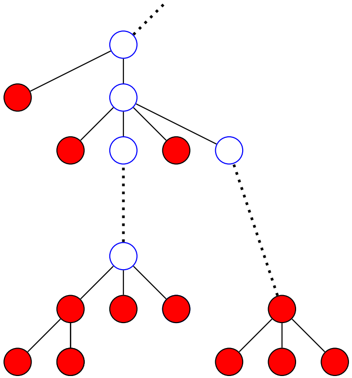
Well-founded Relations



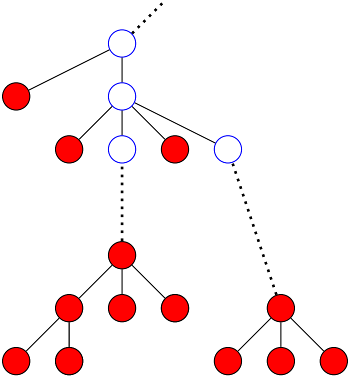
Dotted lines represent any number of elementary relationships



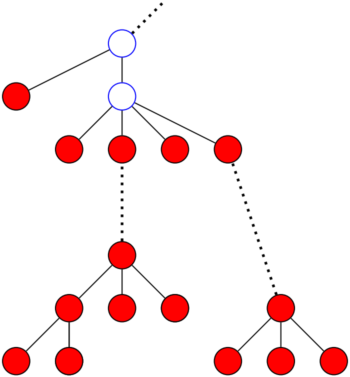
Minimal elements are *accessible*



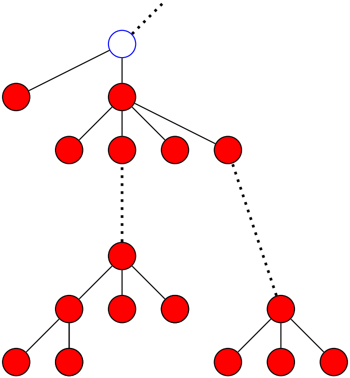
Elements whose all predecessors are accessible become accessible

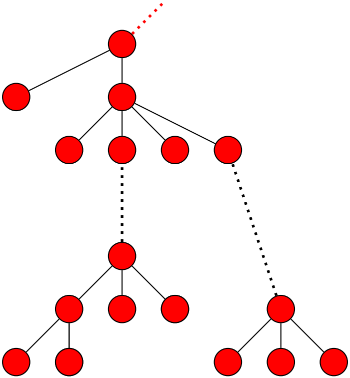


. . .



Some time later . . .





Well founded relations in Coq

How to encode well founded relations in Coq? By crafting the type of trees with no infinite branch.

Let's try.

Well founded relations in Coq

A type for binary trees:

Well founded relations in Coq

A type for binary trees:

```
Inductive btree : Type :=
```

Well founded relations in Coq

A type for binary trees:

```
Inductive btree : Type :=
  | Leaf : btree
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Well founded relations in Coq

A type for binary trees:

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Well founded relations in Coq

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A type for finitly branching trees:

Well founded relations in Coq

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Inductive btree : Type :=
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A type for finitly branching trees:

```
Inductive ntree : Type :=
   | Leaf : ntree
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Well founded relations in Coq

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   | Node :
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Well founded relations in Coq

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Inductive ntree : Type :=
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Well founded relations in Coq

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A type for finitly branching trees:

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Inductive ntree : Type :=
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   | Node :  (list ntree) -> ntree.
```

A type for countably branching trees:

Well founded relations in Coq

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  | Node : btree -> btree -> btree.
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A type for finitly branching trees:

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Inductive ntree : Type :=
   | Leaf : ntree
   | Node :  (list ntree) -> ntree.
```

A type for countably branching trees:

```
Inductive itree : Type :=
    | Leaf : itree
    | Node
```

Well founded relations in Coq

A type for binary trees:

```
Inductive btree : Type :=
  | Leaf : btree
  | Node : btree -> btree -> btree.
```

A type for finitly branching trees:

```
Inductive ntree : Type :=
   | Leaf : ntree
   | Node :  (list ntree) -> ntree.
```

A type for countably branching trees:

```
Inductive itree : Type :=
    | Leaf : itree
    | Node  : (nat -> itree) -> itree.
```

Well founded relations in Coq

A type for countably branching trees:

```
Inductive itree : Type :=
    | Leaf : itree
    | Node : (nat -> itree) -> itree.
```

Well founded relations in Coq

A type for countably branching trees:

```
Inductive itree : Type :=
    | Leaf : itree
    | Node : (nat -> itree) -> itree.
```

We can still program with inhabitants of that type:

```
Fixpoint ileft t :=
 match t with
 | ILeaf => t
 | INode f => ileft (f 0)
 end.
```

Well founded relations in Coq

A (dependent) type for trees with bounded degree:

```
Inductive dtree' : nat -> Type :=
```

Well founded relations in Coq

A (dependent) type for trees with bounded degree:

```
Inductive dtree' : nat -> Type :=
    | Leaf' : forall n, dtree' n
```

Well founded relations in Coq

A (dependent) type for trees with bounded degree:

```
Inductive dtree' : nat -> Type :=
    | Leaf' : forall n, dtree' n
    | Node'
```

Well founded relations in Coq

A (dependent) type for trees with bounded degree:

```
Inductive dtree' : nat -> Type :=
    | Leaf' : forall n, dtree' n
    | Node'  : forall n,
           (forall m, m < n -> dtree' m) -> dtree' n.
```

Well founded relations in Coq

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```
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    | Leaf' : forall n, dtree' n
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In fact the `Leaf` constructor can be removed:

Well founded relations in Coq

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```
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    | Node
```

Well founded relations in Coq

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In fact the `Leaf` constructor can be removed:

```
Inductive dtree : nat -> Type :=
    | Node  : forall n,
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```

Because we can construct a `(DLeaf : dtree 0)` (exercise).

Well founded relations in Coq

A (dependent) type for trees with bounded degree:

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Well founded relations in Coq

A (dependent) type for trees with bounded degree:

```
Inductive dtree : nat -> Type :=
    | Node : forall n,
             (forall m, m < n -> dtree m) -> dtree n.
```

We can generalize to a binary relation `R` on `nat`:

```
Inductive atree (R : nat -> nat -> Prop) : nat -> Type :=
| ANode : forall n,
             (forall m, R m n -> atree R m) -> atree R n.
```

Well founded relations in Coq

A (dependent) type for trees with bounded degree:

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            (forall m, R m n -> atree R m) -> atree R n.
```

If it satisfies the following, this relation is for sure well founded:

```
Definition nat_well_founded (R : nat -> nat -> Prop) :=
    forall n, atree R n.
```

Well founded relations in Coq

A relation is well founded if all elements are accessible.

```
Inductive Acc (A : Type) (R : A->A->Prop) (x:A) : Prop :=
  Acc_intro :
    (forall y : A, R y x -> Acc R y) -> Acc R x.

Definition well_founded (A:Type) (R:A->A->Prop) :=
  forall a, Acc R a.
```

It is possible to define functions by recursion on the accessibility proof of an element (`Function`, `Program` are based on this).

Proving that some relation is well-founded

Coq's Standard Library provides us with some useful examples of well-founded relations :

- The predicate `lt` over `nat` (but you can use measure instead)
- The predicate `Zwf` c, which is the restriction of $<$ to the interval $[c, \infty[$ of $\mathbb{Z}$.

Libraries `Relations`, `Wellfounded` contains (dependent) cartesian product, transitive closure, lexicographic product and exponentiation.

More examples: log10

```
Function log10 (n : Z) {wf (Zwf 1) n} : Z :=
  if Zlt_bool n 10 then 0 else 1 + log10 (n / 10).
Proof.
  (* first goal *)
  intros n Hleb.
  unfold Zwf.
  generalize (Zlt_cases n 10) (Z_div_lt n 10);rewrite Hleb.
  omega.
  (* Second goal *)
  apply Zwf_well_founded.
Defined.
(* Compute log10 2. : and wait (for a long time) ... *)
```

log10 can also be defined using measure

```
Function log10 (n : Z) {measure Zabs_nat n} : Z :=
  if Zlt_bool n 10 then 0 else 1 + log10 (n / 10).
Proof.
  (* first goal *)
  intros n Hleb.
  unfold Zwf;generalize (Zlt_cases n 10); rewrite Hleb;intro
  apply Zabs_nat_lt.
  split.
  apply Z_div_pos;omega.
  apply Zdiv_lt_upper_bound;omega.
Defined.
```

Generating one's own induction principle

Sometime, the generated induction principle is not usable.

```
Inductive tree (A:Type) :=
 | Node : A -> list (tree A) -> tree A.

Check tree_ind.

tree_ind
  : forall (A : Type) (P : tree A -> Prop),
       (forall (a : A) (l : list (tree A)), P (Node A a l))
       forall t : tree A, P t
```

Generating one's own induction principle

```
my_tree_ind : forall (A : Type)
  (P : tree A -> Prop) (Pl : list (tree A) -> Prop),
  (forall a l, Pl l -> P (Node _ a l)) ->
  Pl nil ->
  (forall t l, P t -> Pl l -> Pl (t :: l)) ->
  forall t, P t
```

This is a good exercise...

Principles of coinductive definitions

- Type (or family of types) defined by its constructors
- Values (closed normal term) begins with a constructor
  Construction by pattern-matching (`match...with...end`)
- Biggest fixpoint $\nu X.F\,X$: infinite objects
  - Co-iteration: $\forall X, (X \subseteq FX) \to X \subseteq \nu X.F\,X$
  - Co-recursion: $\forall X, (X \subseteq F\,(X + \nu X.FX)) \to X \subseteq \nu X.FX$
  - Co-fixpoint: $f := H(f) : \nu X.FX$
    Recursive calls on f are guarded by the constructors of $\nu X.FX$.

Example: streams

```
Variable A : Type.

CoInductive Stream : Type :=
  Cons : A -> Stream -> Stream.

Definition hd (s:Stream) : A
  := match s with Cons a _ => a end.

Definition tl (s:Stream) : Stream
  := match s with Cons a t => t end.
```

Example: streams

```
Variable A : Type.

CoInductive Stream : Type :=
  Cons : A -> Stream -> Stream.

CoFixpoint cte (a:A) := Cons a (cte a).

Lemma cte_hd : forall a, hd (cte a) = a.
Proof. trivial. Qed.

Lemma cte_tl : forall a, tl (cte a) = cte a.
Proof. trivial. Qed.

Lemma cte_eq : forall a, cte a = Cons a (cte a).
  Proof.
  intros.
  transitivity (Cons (hd (cte a)) (tl (cte a)));
  trivial.
  now case (cte a); auto.
Qed.
```

Functions should also be guarded

Filter on stream

```
Variable p:A->bool.
CoFixpoint filter (s:Stream) : Stream :=
  if p (hd s) then Cons (hd s) (filter (tl s))
  else filter (tl s)
```

Might introduce a closed term of type `Stream` which does not reduce to a constructor.

Coinductive family

Notion of infinite proof:

```
CoFixpoint cte2 (a:A) := Cons a (Cons a (cte2 a)).
```

How to prove $\mathrm{cte}\, a = \mathrm{cte2}\, a$?
Definition of an extentional (bisimulation) equality predicate:

```
CoInductive eqS (s t:Stream) : Prop :=
   eqS_intros : hd s = hd t -> eqS (tl s) (tl t)
        -> eqS s t.
```

Proof

```
CoFixpoint cte_p1 a : eqS (cte a)  (cte2 a) :=
   eqS_intro (refl a) (cte_p2 a)
with cte_p2 a : eqS (cte a) (Cons a (cte2 a)) :=
   eqS_intro (refl a) (cte_p1 a).
```

A CS example

The computation monad (Megacz – PLPV'07, . . .):

```
CoInductive comp (A : Type) :=
| Done (a : A) : comp A
| Step (c : comp A) : comp A
```

One `Step` is one "tick" of a computation.
Exercise: Show it is a monad, with special action:

```
eval : forall A, comp A -> nat -> option A
```

What's the right notion of equality on computations?
Write the Collatz function using this monad:
`http://en.wikipedia.org/wiki/Collatz_conjecture`.